Route to Chaos in the $^{15}NH_3$ far infrared ring laser

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Abstract

The route to chaos as the pump power is raised in the Lorenz-like $^{15}NH_3$ laser is studied using transients and compared with the route which is expected in the Lorenz Haken equations. The differences between the routes to chaos, as normally displayed by the experimental system and the Lorenz Haken equations, is described in terms of the bifurcations experienced in each system. It is also shown that the experimental system has a small parameter regime where the route to chaos is the same as in the Lorenz Haken equations.

1. INTRODUCTION

In 1963 Lorenz[1] reduced the equations for convective fluid flow into three first order coupled nonlinear differential equations and demonstrated with these the idea of sensitive dependence upon initial conditions and chaos. It was shown by Haken in 1975 that these equations are isomorphic with the Maxwell Bloch laser equations for a homogeneously broadened single mode travelling wave resonantly tuned laser in which the modes are presumed plane waves and the pumping is presumed uniform[2]. Even though these equations are isomorphic their conventional layout is different, as shown in equations 1-3 and 4-6.

$$\dot{x} = \sigma[y - x] \tag{1}$$
$$\dot{y} = \rho x - y - xz \tag{2}$$
$$\dot{z} = xy - \beta z \tag{3}$$

$$\dot{E} = \kappa P - \kappa E \tag{4}$$
$$\dot{P} = \gamma_\perp[ED - P] \tag{5}$$
$$\dot{D} = \gamma_\parallel[\lambda - 1 - D - \lambda EP] \tag{6}$$

The variables in the Lorenz equations, namely x,y and z correspond to the slowly varying amplitudes of the electric field E and polarization P and the inversion D respectively in the Lorenz-Haken equations. The parameters are related via $\beta = \frac{\gamma_\parallel}{\gamma_\perp}$, $\sigma = \frac{\kappa}{\gamma_\perp}$ and $\rho = \lambda + 1$, where $\gamma_\parallel$ is the relaxation rate of the inversion, $\gamma_\perp$ is the relaxation rate of the polarization,κ is the field relaxation rate, and λ represents the pump power, normalized to the threshold for CW lasing.

It was conjectured that Lorenz-like chaotic behaviour could exist in an optically pumped far infrared ring laser. Subsequently, such chaos, characterised by variable length trains of pulses of increasing amplitude ("spirals") was observed by Weiss and coworkers in NH_3 lasers, under appropriate conditions[3]. In this paper we will not discuss the other sorts of chaos exhibited by the laser while it is under other conditions. Detailed study has since shown that there are only a few minor differences between the laser's Lorenz-like chaos and Lorenz Haken chaos[4]. One of the most obvious differences is the appearance of an anomalously large peak nearly always observed at the start of every spiral in the laser[5]. This has been attributed to deviations from the simple model due to Doppler broadening or coherent pumping effects on the pump transition[6]. It was also found that this effect appears if the theory simply accounts for the fact that the laser and the pump have transverse structures (i.e. there exist real modes in the cavity) and are not plane wave as the Lorenz Haken equations assume[7]. Perhaps it is a generic effect of any small deviation from the Lorenz equations. The other obvious difference in the behaviour of the laser is that under nearly all circumstances the laser pulses periodically below chaos threshold and does not make a direct transition from the steady state to chaos as the Lorenz Haken equations predict. Although topological arguments suggest that in any continuous transition from a fixed point to a strange attractor the attractor should become one-dimensional, corresponding to a periodic orbit, at some point, the transition to chaos in the Lorenz equations is a global bifurcation with no intermediate states. Further, the model accounting for nonuniformity of pump and electric field also shows a transition with no intermediate periodic state. The aim of this paper is to describe the route to Lorenz like chaos in an optically pumped far infrared ammonia laser and suggest the possible nature of the bifurcations and hence the underlying topology of the laser attractor as the pump power is increased from the lasing threshold to chaos.

The next section will give a short revision of the route to chaos with increasing pump power for the Lorenz Haken equations to allow easy comparison with the route to chaos observed in the laser, which will be described in section 3. In section 4 the results of section 3 will be extended to include the effect of detuning in the laser.

2. THE LORENZ HAKEN EQUATIONS

The Lorenz equations have been studied extensively over the years as a basic illustrator of chaos and demonstration platform for various theories relating to chaos. An extensive review of Lorenz chaos was undertaken by Sparrow[8], who subsequently showed numerically that there is considerable complexity in the dynamics of the Lorenz equations which was not previously analytically explained. This included such phenomena as homoclinic explosions which give rise to stable periodic motion, and a heteroclinic bifurcation.

To explain how the route to chaos in the laser differs from that of the equations a brief revision of the bifurcations in the equations will be given. Below lasing threshold the origin is globally attracting. That is, all trajectories finish at the origin if $0 < \lambda + 1 < 1$. At $\lambda > 0$ a simple bifurcation occurs which creates two stationary points at $(\pm\sqrt{\frac{\gamma_\perp \lambda}{\gamma_\parallel}}, \pm\sqrt{\frac{\gamma_\perp \lambda}{\gamma_\parallel}}, \lambda)$ henceforth called C_1 and C_2. At this bifurcation the origin loses its stability, however C_1 and C_2 are stable. The two points represent the steady state solutions for the system in the two

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possible orientations that they might find themselves, with the elctric field and polarization differing by a $/Pi$ phase shift. At $\lambda + 1 = \frac{\kappa}{\gamma_\perp}(\frac{\kappa + \gamma_\parallel + 3\gamma_\perp}{\kappa - \gamma_\parallel - \gamma_\perp})$ the equations go through a second bifurcation in which the points C_1 and C_2 lose their stability. The bifurcation is of the type known as a Hopf bifurcation, which may occur in two ways. The bifurcation is "supercritical" if each point loses its stability by expelling a stable periodic orbit. It is "subcritical" if they lose their stability by absorbing an unstable periodic orbit. For values of λ greater than the Hopf bifurcation point the origin and both C_1 and C_2 are all unstable. There is another point on the route to chaos which delineates an important event and this is the occurence of a homoclinic orbit. The homoclinic orbit separates the region between normal stable behaviour and possible metastable chaos, associated with an unstable periodic orbit. This can be understood by considering operation just above lasing threshold. A trajectory started very near to the origin will move out and, as shown in figure 1, quickly spiral into either C_1 or C_2 depending on which the point started closer to. As λ is increased the trajectories around the C_i widen until the trajectory in both the forward and backward propagation of time moves towards the origin. There is therefore a homoclinic orbit associated with the origin. If λ is increased slightly further the trajectory jumps across to spiral into the opposite stable point. Figure 1 illustrates this point and Sparrow[8] gives a more detailed explanation of how this occurs. Increasing λ above the homoclinic explosion, as shown by Sparrow, the homoclinic orbit becomes an unstable periodic orbit, getting smaller and finally getting absorbed by the stable point C_1 or C_2 at the Hopf bifurcation.

The implications of the existence of these orbits is best described by figure 2 where the bifurcations are placed into context with each other. When the unstable periodic orbits decrease in size there is a possibilty for the trajectory to oscillate between the two sides of the attractor and do so for long periods of time. However because there are two stable attractors embedded in an unstable attractor the trajectory must finally intersect the stable manifolds and spiral into the stable points C_1 or C_2. So instead of the infinitely long single trajectory as experienced in chaos there are an infinite number of finite length trajectories. This is generally called "preturbulence" or "metastable chaos" and refers to the fact that the dynamics may look chaotic, but will at some finite time intersect the stable manifold and spiral toward one of the stable points.

3. ROUTE TO CHAOS IN THE LASER

The experimental setup we use has been fully described by Tin Win et al[9]. Briefly, it is a $^{15}NH_3$ far infrared ring laser which is pumped by a $^{13}CO_2$ laser whose pump output is attenuated by an acousto optic modulator (AOM) of rise time 0.8 μs. To gain more insight into the attractor properties, the laser was switched from above chaos threshold to below chaos threshold periodically by the AOM. Each period above chaos was about 50-100 chaotic pulses long, whereas the period below chaos threshold was generally about 100-150 pulses in length. The time between pulses is about 0.8 μs. Figure 3 illustrates the time series of the Lorenz Haken equations as a result of such a sudden decrease in λ from just above chaos threshold to below chaos threshold. The transition to the new state involves two phases. Firstly there is a variable length period of chaotic pulsations, followed by a damped oscillation. Theoretical studies of this behaviour[10] have shown that the length of the chaotic

phase can be characterised by an exponential distribution with a mean length dependent on how far below chaos threshold the pump power is. A detailed experimental study of this behaviour is being undertaken with the results to be published elsewhere[11]. Figure 4 shows a typical time series for the $^{15}NH_3$ far infrared laser, demonstrating the predicted behaviour. It should be noted that the only variable which is measurable in the laser is the intensity. This of course places some restrictions on the information which is processable from any time series obtained. Comparison of return maps and statistical analysis[12] have shown that the chaos exhibited by the laser, for specific parameters is very similar to Lorenz chaos. Because the chaos is Lorenz-like it can be assumed that the dynamics above chaos threshold must have a topologically similar structure to the Lorenz Haken equations. At the other extreme, Tin Win et al[9] determined the parameters $\gamma_\parallel$, $\gamma_\perp$ and κ for the $^{15}NH_3$ laser by comparing the decay envelopes for transients for low pump power with the linearised Lorenz Haken equations. These studies showed that the laser operates in reasonably close agreement with the Lorenz Haken equations for low pump powers.

Figure 3 and figure 4 compare the dynamics of the experiment with what is expected from the equations. In the parameter range where the dynamics of the laser exhibits preturbulence its behaviour also seems to be very similar to the Lorenz Haken equations[11]. In the Lorenz Haken equations, close to but above the homoclinic explosion a trajectory started in the vicinity of the origin should go around one of the points C_i once and then spiral into the other point. It is normally possible to measure only the intensity in these experiments which obviously makes it difficult to observe a jump from one side of the attractor to the other. One way to detect a jump is to measure the phase of the laser radiation by heterodyne detection. We have carried out this experiment using a second $^{15}NH_3$ laser (a standing wave laser operated continuously with a few milliwatts of output power) as a local oscillator and a Schottky barrier detector as mixer with the two beams combined at a polyethylene beamsplitter. The output from the detector contains a homodyne spectrum from the ring laser, extending out several megahertz, and the heterodyne spectrum centred on the mean frequency difference between the two lasers which was limited to about 6 MHz. The detector signal was digitised, the homodyne spectrum removed numerically and quadrature field components (or amplitude and phase) reconstructed by numerically mixing with a signal at the mean difference frequency, in analogy with RF engineering methods. Appendix A has a thorough explanation of the method used to extract the amplitude and phase of the signal. This method is similar to that used by Tang et al[14], but handles transients better. Examples of the results are shown in figures 5 and 6. In both cases the laser pump power was rapidly switched on from zero to a level giving CW output, and the transient approach to the steady state was recorded. Figure 5 shows the intensity and phase variation at low pump powers with an apparent slow drift in phase (a result of a drift in local oscillator frequency relative to the ring laser) but no jump in phase. Figure 6, however shows the intensity and phase variation with the pump power above the first homoclinic explosion, with a π phase jump clearly present between the first and second pulses.

The only difference between the laser and the equations is the occurence of a stable oscillation within the preturbulent regime in the laser. Instead of finally decaying to a steady value corresponding to a fixed point, the decay is to a stable oscillation. This oscillation is also met if the pump power is raised slowly in a quasi-static manner, with a threshold somewhat below the chaos threshold. The oscillation starts small at lower pump powers

and grows with increasing pump power until the laser shows no more preturbulence. When the stable oscillation grows to be about the same size as the preturbulence, the laser jumps into chaos and stays there. Figure 7 shows a typical graph of percentage pump power below chaos threshold and oscillation amplitude for a particular set of parameters. Figure 4 shows an example of such behaviour. When this is compared with the time series of figure 3 it is obvious that this does not occur in the Lorenz Haken equations.

To revise what happens as the pump power is increased;

1.   For pump powers below 120 mW at a pressure of 38 μbar there is no lasing

2.   For powers greater than this the laser makes a transient oscillation which damps to steady state much like the Lorenz Haken equations, moving towards one or other of the stable fixed points (Figure 5).

3.   A point started near the origin and close to Lorenz-like chaos will exhibit a jump from one side of the attractor to the other and subsequently collapse into steady state lasing (Figure 6).

4.   For slightly higher pump powers the laser exhibits preturbulence much like that of the Lorenz Haken equations. The differences between them lies in the appearance of a stable oscillation in the laser which grows in amplitude as the pump power increases (Figures 3 and 4).

So far the dynamics of the Lorenz Haken equations and the dynamics of the laser have been described to explain their respective routes to chaos as the pump power is increased. Both routes look quite similar except for the appearance of the stable oscillation in the laser. A possible explanation for the occurrence of such a stable oscillation is shown in figure 8. Starting at low pump power the first significant event, as pump power is increased, is the appearance of the two unstable periodic orbits around the stable points C_i as is the case with the Lorenz Haken equations. Increasing the pump power further the points C_i bifurcate in a supercritical Hopf bifurcation, each expelling a stable periodic orbit and becoming unstable. A representative diagram showing all the orbits for a specific pump power is shown in figure 9. Increasing the pump power from here on results in an increasing size of the stable homoclinic orbits. Chaos is reached when the unstable and stable homoclinic orbits annihilate each other to leave Lorenz like chaos. The most trivial implication of this route to chaos is that the points C_i go unstable at lower pump powers than they do in the Lorenz Haken equations. This also has the implication that the range of pump powers over which preturbulence exists may be reduced.

4. THE EFFECTS OF DETUNING

During the course of the experiment it was observed that the amplitude of the stable oscillations during preturbulent chaos is dependent upon detuning of the laser cavity. It was found that within an extremely small range of detuning, preturbulence occurred with the laser relaxing down to steady state rather than the stable oscillations. This range was far narrower than the range over which Lorenz like chaos was observed, and so narrow that it was not possible to carry out an exhaustive investigation. However, figure 10 shows an intensity time series of such an event. At the moment it is not possible with our setup to measure how far from zero detuning the laser is, no conclusions can therefore be made as to where this region resides. It would not be surprising however to find this region at zero

detuning. This would therefore mean that there does exist a regime where the laser chaos corresponds exactly to the Lorenz laser case described by the Lorenz Haken equations. Such a regime has also been observed in the $^{14}NH_3$ laser[15].

5. CONCLUSIONS

It has been shown, based on measurements of the transient behaviour of the laser, that the $^{15}NH_3$ far infrared laser has a route to chaos with increasing pump power which is slightly different from that of the Lorenz Haken equations. As with any other laser it has a minimum pump power below which no lasing is possible. For powers greater than this the laser makes a transient oscillation which damps to steady state much like the Lorenz Haken equations. Using heterodyne techniques to see the field there was found to be a pump power above which the laser exhibits a jump from one side of the attractor to the other where the field subsequently collapsed to steady state lasing. At slightly higher pump powers the laser was found to exhibit preturbulence leading to a stable oscillation which grows in amplitude with increasing pump power. Finally, above a certain power Lorenz like chaos is observed.

The route to chaos, as observed in our laser, has been conjectured to involve an extra stable periodic orbit which annihilates with the unstable periodic orbit to leave Lorenz like chaos.

It was also shown that for an extremely small range of detunings the laser will show preturbulence without the occurence of the stable oscillations. These phenomena will provide an excellent test for improved theoretical models of the laser.

6. Appendix A

In the heterodyne measurement of the chaotic laser, the signal from the chaotic laser and the signal from the local oscillator are coupled together by a beam splitter. The mixed signals are then detected by a Schottky barrier diode. The electric field of the chaotic laser and the local oscillator are:

$$E_1(t) = A_1(t)\sin[\omega_1 t + \phi(t)] \tag{7}$$
$$E_2(t) = A_2\sin(\omega_2 t) \tag{8}$$

where A_1, ω_1 and ϕ are the amplitude, the "optical" frequency and the phase of the electrical field of the ring laser respectively. A_2 and ω_2 are the amplitude and the "optical" frequency of the local oscillator respectively. A_1 and ϕ are time dependent. A_2, ω_1 and ω_2 are constant.

The response of the Schottky diode is[16]

$$I(t) \propto e^{\alpha V} = e^{\beta(E_1+E_2)} = 1 + \beta(E_1 + E_2) + \frac{1}{2}\beta^2(E_1 + E_2)^2 + \ldots \tag{9}$$

where α and β are constants.

Because the rest of the electronics responds only to frequencies much lower than the "optical" frequencies ω_1 and ω_2, and a bias voltage is applied to the Schottky diode to make it favor the square term, in the output from the Schottky detector only the square term

is important. So for $\Delta t \gg 1/\omega_1, 1/\omega_2$ (Δt is the response time of the Schottky diode) the output from the Schottky diode is

$$
\begin{aligned}
< I >_{\Delta t} &\propto < (E_1 + E_2)^2 > \\
&= < A_1^2 \sin^2(\omega_1 t + \phi) + A_2^2 \sin^2(\omega_2 t) + 2A_1 A_2 \sin(\omega_1 t + \phi) \sin(\omega_2 t) > \\
&\propto \underbrace{A_1^2 + A_2^2}_{homodyne\ part} + \underbrace{2A_1 A_2 \cos[(\omega_1 - \omega_2)t + \phi]}_{heterodyne\ part}
\end{aligned} \tag{10}
$$

The output from the Schottky diode is a mixture of the desired heterodyne signal carried by the beat between the chaotic laser and the local oscillator with an unwanted homodyne component. The time variation of A_1 and ϕ lead to a finite width for both the homodyne and heterodyne spectra in the frequency domain. It was ensured that the spectra did not overlap so that the homodyne and heterodyne signals could be separated by using simple filters. If they overlapped a more complicated detection setup, for instance using two Schottky diodes to detect the homodyne signal and the mixed signal respectively, would be necessary.

In the heterodyne signal, A_1 and ϕ are carried by the beat between the chaotic laser and the local oscillator. The carrier signal must be removed from the heterodyne signal in order to reconstruct the chaotic laser field. A Fast Fourier Transformation (FFT) was applied to the experimental record to obtain the signal spectrum and determine the cutoff frequencies for a low and high frequency filter. These filters were subsequently used on the original experimental record to separate the homodyne and heterodyne signals. The homodyne signal was constructed directly from the low pass filtered portion of the signal. The heterodyne time series was separately mixed with a sinusoidal and a cosinusoidal signal of the beat frequency and low pass filtered in order to reconstruct the two orthogonal components of the heterodyne time series and hence the field components and phase.

Tests on synthetic data show this method is reliable for in analyzing transient signals, as it is able to reconstruct the original field amplitude and phase perfectly. By contrast, the signal processing method used by Tang et al[14], which is based on maniopulation of Fourier transforms, is able to reconstruct periodic data reasonably well, but it introduces errors in the amplitude and the phase when the signals are transient.

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Figure Captions

Figure 1: Trajectories of the Lorenz Haken laser equations at pump powers, λ equal to 1.0 and 2.0 below the first homoclinic explosion, $\approx$3.0351841193 for the homoclinic orbit, and 4.0 above the homoclinic explosion. Parameter values of $\gamma_\parallel = 1/8$, $\gamma_\perp = 1/2$ and $\kappa = 1$ are appropriate for the $^{15}NH_3$ laser and give a chaos threshold of 13. The circles indicate the stable points towards which the trajectories will gravitate.

Figure 2: The bifurcation diagram for the Lorenz Haken laser equations. The $x = y$ axis symbolises the symmetry along that plane of the equations.

Figure 3: Intensity time trace of preturbulence in the Lorenz Haken laser equations at $\gamma_\parallel = 1/8$, $\gamma_\perp = 1/2$ and $\kappa = 1$ resulting from the fast switch down from $\lambda = 14$ above the chaos threshold to $\lambda = 10$ below the chaos threshold.

Figure 4: Typical intensity time trace of preturbulence in the $^{15}NH_3$ laser as the laser is switched from slightly above the 5.85 W chaos threshold to a factor of 0.1 below chaos threshold.

Figure 5: Phase and intensity evolution of the $^{15}NH_3$ laser. The dashed line shows the decaying pulses of the intensity and the solid line the phase of the light. The phase shows a slow drift as the pulses decay indicating the absence of a jump from one side of the attractor ot the other. The pump power was a factor of 0.7 below chaos threshold.

Figure 6: Phase and intensity evolution of the $^{15}NH_3$ laser at switch on. The dashed line shows the decaying pulses of the intensity and the solid line the phase of the light. The phase shows a distinctive jump of π after the first pulse to indicate a jump from one side of the attractor to the other. The pump power was a factor of 0.5 below chaos threshold.

Figure 7: Amplitude of the intensity oscillations observed in the $^{15}NH_3$ laser as a function of pump power.

Figure 8: The bifurcation diagram of the $^{15}NH_3$ laser. The difference between this diagram and that for the Lorenz Haken equations shown in figure 2 is the presence of a supercritical Hopf bifurcation within, and which finally annihilates, the already existing homoclinic orbit.

Figure 9: Representation of the stable and unstable orbits relative to their respective fixed points. The format of the figure is as for figure 2.

Figure 10: Preturbulence in the the $^{15}NH_3$ laser for the detuning range where the preturbulence settles down to a steady state. The laser is switched from slightly above the 5.85 W chaos threshold to a factor of 0.05 below chaos threshold.

x=y

homoclinic
explosion

unstable
periodic orbits

pitchfork
bifurcation

sub-critical
Hopf bifurcation

0.0

λ

stable
attractor

possible metastable chaos

chaos

11

E=P

homoclinic
explosion

super-critical
Hopf bifurcation

stable
periodic orbits

pitchfork
bifurcation

unstable
periodic orbits

0.0

λ

stable
attractor

possible metastable
chaos

chaos

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